Ancient Chinese Mathematics:

Right Triangles & Their Applications

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Mathematics is often thought of as a purely European development. History of mathematics books often focus on Greece as the epicenter for early mathematical discoveries. For example, the main theorem about right triangles (the sum of the squares of the two smaller sides of a right triangle equals the square of the hypotenuse) is attributed to and now bears the name of a Greek, Pythagoras of Samos, born around 570-560 BC. Pythagoras is often credited with the first proof of the theorem; however his actual written proof has not been found. Earlier civilizations definitely knew about this geometric fact, and perhaps after his travels, Pythagoras took this information back to Greece. Around the same time as Pythagoras, the *Sulvasūtras* by Apastamba demonstrates that the Indian civilization was familiar the Pythagorean Theorem and Pythagorean Triples (sets of three numbers that satisfy the Pythagorean Theorem) (Boyer 207). The Babylonians discovered the Triples much earlier. In the Babylonian tablet known as Plimpton 322, Pythagorean Triples are arranged so that the first row corresponds to the ratio c^2/b^2, the second to the number a, and the third to the number c, such that $a^2 + b^2 = c^2$. This tablet is thought to come from approximately 1900-1600 BC, long before Pythagoras' time (Boyer 34-37). While the Babylonians certainly knew about Triples very early on, the Chinese may have proven the Pythagorean Theorem the earliest; some estimates are as early as 1100 BC, although sixth century BC is more generally accepted (Swetz and Kao 14). The Pythagorean Theorem and right-angled triangles were very prominent in Chinese writings, both in mathematical treatises and in more practical science books. The Chinese grasped many right-angled triangle principles early on, and applied them to practical problems.

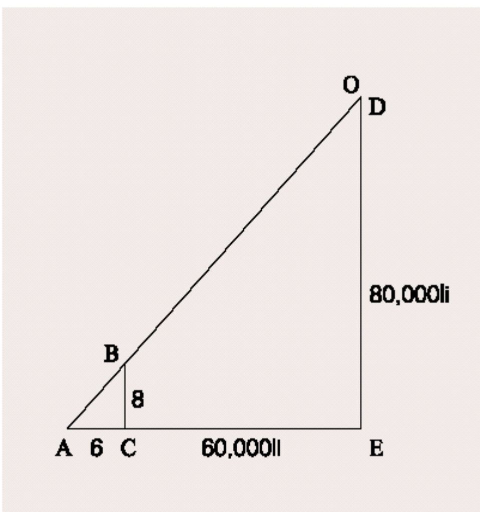
China is an ancient civilization, akin to Babylon and Egypt. Like the notched wolf bone in Europe, the first signs of math in China are the markings on tortoise shells and cattle bones, known as oracle bones, from the Shang dynasty, approximately 1200 BC. Later, the Chinese did calculations using small bamboo counting rods, which lead to the use of rod numerals and a positional system for writing numbers. Unlike the Babylonians and their sexagesimal system, the Chinese tended to decimalize fractions, and unlike the Egyptians and their unit fractions, the Chinese used common fractions and were able to find the lowest common denominator in order to add different fractions. The Chinese were also comfortable with negative numbers, using a red set of counting rods for positive numbers and a black set of counting rods for negative numbers, similar to modern accounting except with the colors are reversed (Boyer 198-201). Like early writings in Egypt and Babylon, ancient Chinese mathematic books tended to be collections of practical problems, giving the problem first, then the answer, and sometimes the solution method. The earliest books on mathematics seem to have been written in the Han dynasty, although dating the books is very difficult as the emperor Shih Huang-ti of the Ch'in dynasty ordered a burning of books in 213 BC. In the following Han dynasty, mathematicians had to rewrite all of the old books from memory or from hidden scrolls (Swetz and Kao 17). The early books are thus thought not to be the work of any one mathematician, but rather a collection of the mathematical knowledge up to that age. The main early writings include the *Zhou bi suan jing* (*The Arithmetic Classic of the Gnomon and the Circular Paths of Heaven)*, which is sometimes written as *Chou Pei Suan Ching*; and the *Chiu chang suan shi* (*The Nine Chapters on the Mathematical Art)*, which is sometimes written as *Jiu zhang suan shu*. Both books were commented on by many later mathematicians with Liu Hui, an official in the Wei kingdom, being the first to comment on the *Nine Chapters* (Swetz 14). In addition to remarking on the existing work, Liu Hui also added nine problems to the end of the *Nine Chapters*; this addition was later published as the *Haidao Suanjing* (*The Sea Island Mathematical Manual*). The added nine problems, as well as the last chapter of the *Nine Chapters* and several discussions in the *Zhou bi*, all involved the application of right-angled triangles and demonstrated the Chinese knowledge of the Pythagorean Theorem.

The *Zhou bi suan jing*, an ancient Chinese text on astronomy and mathematics, is often referred to as *The Arithmetic Classic of the Gnomon and the Circular Paths of Heaven,* where the gnomon is an upright pole or post used for astronomical observations (Cullen 238). The *Zhou bi* is thought to have been assembled in the Han dynasty, but was probably first written in the preceding Zhou dynasty. It was later commented on by three main mathematicians: Zhao Shang (3rd century AD), Zhen Luan (6th century AD), and Li Chunfeng (7th century AD) (Cullen 68-73). The book is written not just as example problems and answers, but as a dialogue usually between a master and a student. The original text is very basic as the master assigns complex computations to the student without any indications of how to solve the problems. Details and explanations are later given by the commentators. The *Zhou bi* uses right-angled triangles to explain astronomy and is often interpreted as including a proof of the Pythagorean Theorem.

The astronomy of the *Zhou bi* follows gai tian cosmology, which says that the earth is a flat plane and the heavens rotate above the earth (Cullen xi). With this idea, the height of the sun and other heavenly bodies can be found using a gnomon, or bi, and its shadow. When the sun is directly overhead, the gnomon casts no shadow, but as the gnomon is moved north or south, the shadow changes length. In the second section of the *Zhao bi*, Chen Zi, the master, is speaking to Rong Fang, his student. He states (Cullen 178):

> The zhou bi is eight chi in length. On the day of the summer solstice its shadow is one chi and six cun. The bi is the altitude, and the exact shadow is the base. 1000 li due south the base is one chi and five cun, and 1000 li due north the base if one chi and seven cun. The further south the sun is, the longer the shadow.

This idea is referred to as the shadow principle: that for every 1000 li away from the 'no shadow' spot, the shadow of an eight chi gnomon increases by one cun. One chi equals 10 cun, and one li equals 1800 chi (Swetz 19). The next passage states:



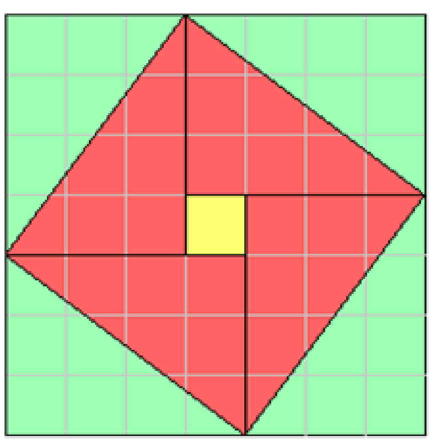
> Wait until the base is six chi, then take a bamboo … of length eight chi. … So start from the base, and take the bi as the altitude. 60,000 li from the bi, at the subsolar point a bi casts no shadow. From this point up to the sun is 80,000 li. If we require the oblique distance to the sun, take the subsolar point as the base, and take the height of the sun as the altitude. Square both base and altitude, add them and take the square root, which gives the oblique distance to the sun. The oblique distance to the sun from the position of the bi is 100,000 li.

This illustrates the idea of two similar right-angled triangles; the first triangle (ΔACB) having the gnomon, or bi, as its altitude and the shadow as its base, the second triangle (ΔAED) having the height of the sun as its altitude and the distance from the end of the shadow to the 'no shadow' spot as its base. The text also shows knowledge of the Pythagorean Theorem, squaring the base and the altitude to find the hypotenuse (AD). Two other problems in *Zhou bi* involve finding an unknown side of a right-angled triangle.

These problems entail finding the half chord of a circle, given the radius and the distance from the center to the chord. In the original text, approximate answers to these problems were just given without any explanation of the calculations. When Zhao Shang, the first commentator, wrote out explanations, he used the Pythagorean Theorem, although he doubled all the lengths so as to have a 'nice' answer to the square root. These three problems demonstrate that the ancient Chinese recognized the Pythagorean Theorem and were able to use it in practical applications.

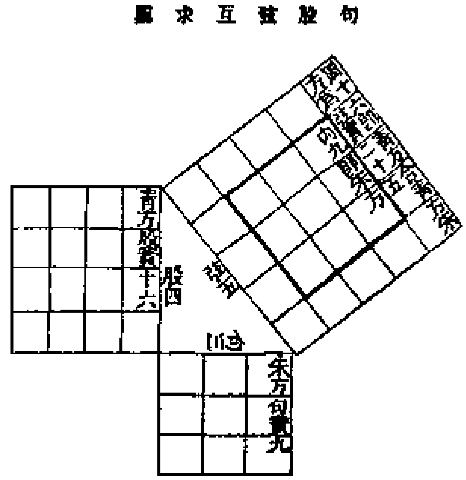
While the Chinese certainly understood the Pythagorean Theorem, were they able to prove it early on? The beginning of the *Zhou bi* is often picked out as a proof, known as hsuan-thu. The original text describes a conversation between Duke of Zhou and Shang Gao, who says (Cullen 174):

> The patterns for these numbers come from the circle and the square. The circle comes from the square, and the square comes from the trysquare, and the trysquare comes from nine nines are eighty-one.
> Therefore fold a trysquare so that the base is three in breadth, the altitude is four in extension, and the diameter is five aslant. Having squared its outside, halve it one trysquare. Placing them round together in a ring, one can form three, four, and five. The two trysquares have a combined length of twenty-five. This is called the accumulation of trysquares.



With just the original text, it's difficult to see the proof. The 3-4-5 right triangle is obvious, but how to use it to prove the Pythagorean Theorem is not obvious. The first commentator, Zhao Shang, added a short essay and a diagram to help explain his interpretation of the proof. The outer square has sides of length 7, and so its area is 49. The green triangles and the red triangles are all right triangles with sides of 3 and 4, and area of 6. The area of the tilted square (red & yellow) must be the area of the total square minus the area of the green triangles; 49 – 4*6 = 25. An area of 25 means that the sides must have length 5, proving that the hypotenuse of the green right triangle must be the square root of 3^2 plus 4^2. This could just be a special case that works. However, Zhao Shang actually explains all of this in his essay without using specific numbers, in a sense doing the proof algebraically with words (Cullen 206-217). If this proof is what is meant by "accumulation of trysquares" in the original text, then the Chinese proved the Pythagorean Theorem most likely before Pythagoras and before Euclid's often cited proof in *Elements*. However, it seems more likely the Zhao Shang's essay is the real proof, in which case the Greeks proved it first.

Around the same time as the *Zhou bi suan jing* was written down, the *Chiu chang suan shu* was also taken down. The *Chiu chang suan shu*, sometimes referred to as *Jiu zhang suan shu*, consists of nine sections devoted to different applications of mathematics, and is often translated as *The Nine Chapters on the Mathematical Art*. Like the *Zhou bi*, the exact author and date of the *Nine Chapters* is not known, although tradition says that Chang Tshang was the first to write the entirety. The *Nine Chapters* were commented on by many mathematicians, including Liu Hui around 250 AD (Swetz and Kao 17). The problems given in the *Nine Chapters* involve many different areas of mathematics. Areas of figures are found, as are square and cube roots. Fractions are added, subtracted, multiplied, and divided using lowest common denominators and greatest common factors. Arithmetic and geometric progressions are introduced, as well as systems of linear equations. The last chapter presents 24 problems on right-angled triangle properties, showing the application of the Pythagorean Theorem (Swetz and Kao 18-24).



In ancient China, the base of a right-angled triangle was called kou or gou (meaning ‘leg’), the altitude was ku or gu (meaning ‘thigh’), and the hypotenuse was hsian or xian (meaning ‘bowstring’) (Cullen 77 & Swetz and Kao 26). The problems in the ninth chapter, called the *Kou-Ku* chapter, progress from easy to hard. The first three problems only require simple knowledge of the 3-4-5 right triangle. In each problem, two of the three sides are given, and the reader is asked to find the third side. In Liu Hui’s commentary, he refers to a diagram similar to one used by Euclid in his *Elements*’ proof, showing the squares of the lengths on the sides of the triangle (Swetz and Kao 26-28).

The rest of the problems in the *Kou-Ku* involve figuring out where the triangle is in the practical situation given and then finding a side of the triangle, given different lengths or ratios. The methods and explanations given clearly make use of the Pythagorean Theorem. In Problem 6, the kou and the difference of hsien and ku are given in the context of a reed plant in the middle of a square pond. The given method to find ku and hsien is to “find the square of [kou], and from it subtract the square of [hsien minus ku]. [Ku] will be equal to the difference divided by twice [hsien minus ku]. To find [hsien] we add [hsien minus ku] to [ku]” (Swetz and Kao 30). Using our modern-day notation of a, b, and c, this method says that $(a^2 - (c-b)^2)/(2(c-b)) = b$. Let’s see:

$$\frac{a^2 - (c-b)^2}{2(c-b)} = \frac{-c^2 + 2bc - b^2 + a^2}{2c - 2b} = \frac{2bc - 2b^2}{2c - 2b} = b \quad \text{when we know } a^2 + b^2 = c^2 \text{ so substitute } a^2 - c^2 = -b^2$$

The word formula given in Problem 6 does indeed find b, or ku.

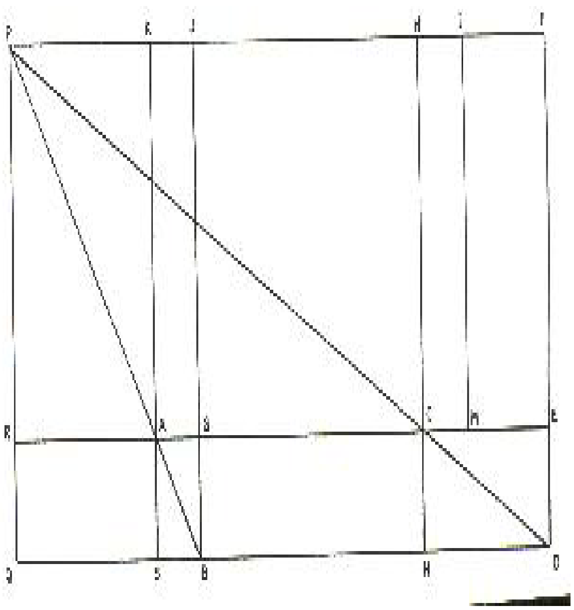
Some of the later problems involve similar triangles and the use of proportions. Problem 17 talks about a 200-paces square city with gates in the center of each side. A tree is located 15 paces from the east gate. If a person goes out the south gate, how many paces will he have to go in order to see the tree? The answer is given as 666 2/3 paces which is "the quotient found by using the 15 paces as a denominator and half the width of the city squared as the numerator" (Swetz and Kao 52). This problem can be thought of as two similar triangles on the east and south sides of the city. The first triangle has one side of 15 paces to the tree and the other side of 100 paces (half the side length of the city). The corresponding sides on the second triangle are the 100 paces (again half the side length of the city) and the unknown number of paces out in order to see the tree. The proportion is:

$\frac{15}{100} = \frac{100}{X}$ so $X = (100)(100)/15 = 666\ 2/3$ (Swetz and Kao 52).

The Chinese understand the ratio similarity between the triangles, and later problems take this idea even further to include quadratic equations of the form $x^2 + ax = b$ (Problems 18 & 19, Swetz and Kao 53-56).

Despite the progress from easy to more difficult in the *Kou-Ku* chapter's 24 problems, even the last eight problems concerning land surveying involve only one simple proportion between similar triangles. These problems make use of one sighting observation in order to figure out some unknown and usually unreachable distance. When Liu Hui commented on the *Nine Chapters*, he added nine more problems to the *Kou-Ku* chapter. These nine problems were an extension of the 24 right-angled triangle problems, and were much more difficult using two or more sighting observations to find two sets of similar triangles (Swetz 27-28). Liu Hui gave the problems, the answers, and the calculations to find the answers; however, his method of proof, if any, is unknown. In the Tang dynasty, the nine problems were taken out of the *Nine Chapters*, and were published as their own mathematical manual, the *Haidao Suanjing*, where Haidao, or Sea Island, refers to the first problem involving finding the distance to an unreachable sea island.

Like the last chapter of the *Nine Chapters*, the problems in *Haidao Suanjing* progress from easiest to hardest, although the difficulty difference between the beginning and the end of Haidao is much less than in the *Kou-Ku* chapter. The first *Haidao* problem, the sea island problem, reads as follows (Swetz 20):



Now for looking at a sea island, erect two poles of the same height, 3 zhang, the distance between the front and rear being a thousand bu. Assume that the read pole is aligned with the front pole. Move away 123 bu from the front pole and observe the peak of the island from ground level; it is seen that the tip of the front pole coincides with the peak. Move backward 127 bu from the rear pole and observe the peak of the island from the ground level again; the tip of the back pole also coincides with the peak. What is the height of the island and how far is it from the pole?

The diagram gives a better understanding of the problem (for clarification, the letters for the top line are P, K, J, H, I, & F; the middle line R, A, G, C, M, & E; and the bottom line Q, S, B, N, & D). AS and CN are the two poles of length 3 zhang, or 30 chi, where one zhang is ten chi and one bu is six chi (Swetz 19). SN is the distance between the poles, 1000 bu, or 6000 chi. SB is the length 123 bu, or 738 chi, and ND is the length 127 bu, or 762 chi. The problem asks for PQ and QS. Using similar triangles, we find that:

$\frac{AS}{PQ} = \frac{SB}{QB}$ & $\frac{CN}{PQ} = \frac{ND}{QD}$ & we're given AS = CN, so $\frac{SB}{QB} = \frac{ND}{QD}$

Using chi lengths given, we have $\frac{738}{738 + QS} = \frac{762}{762 + 6000 + QS}$.

Cross multiplying, we get 562356 + 762*QS = 562356 + 4428000 + 738*QS
24*QS = 4428000
QS = 184500 chi =102 li 150 bu, the answer given in Haidao

To find PQ, we just plug numbers back into one of the proportions:
$\frac{AS}{PQ} = \frac{SB}{QB}$ $\frac{30}{PQ} = \frac{738}{738 + 184500}$.

So 30(185238) = 738*PQ, PQ = 7530 chi = 4 li 55 bu, the answer given in Haidao

Despite the relative simplicity of using similar triangles, after reading Liu Hui's calculation method, some researchers contend that Liu Hui and his Chinese contemporaries did not use similar triangles; instead, it is believed that Liu Hui used a proof method known as "out-in" (Swetz 35-37). The "out-in" principle uses equal area rectangles to find proportions of sides. For the sea island problem, we find the area of QNCR equals the area of CEFH, as the diagonal PD splits the whole rectangle into equal halves and those areas are left over after taking out RCHP and NDEC. Apply the same principle to the big rectangle QBJP, so we get area of QSAR equals area of AGJK, and we place CMIH to have the same area. We observe that area of SNCA is the area of QNCR minus area of QSAR, which is the same as area of CEFH minus area of CMIH, which is MEFI. If area of SNCA equals area of MEFI, then (AS)(SN) = (IM)(ME), but IM = PR, ME = ND - CM, and CM = SB, so (AS)(SN) = PR(ND -SB), giving:

$PR = \frac{(AS)(SN)}{(ND - SB)}$ so $PQ = \frac{(AS)(SN)}{(ND - SB)} + AS$, which is the calculation method given in Haidao

Then we use area of QSAR equals area of CMIH to find QS (Swetz 42-43).

Through the problems and exercises in *Haidao Suanjing*, as well as in the *Kou-Ku* chapter of the *Nine Chapters* and in *Zhoi bi*, the ancient Chinese use of the Pythagorean Theorem and other properties of right-angled triangles is obvious. Unfortunately, the difficulty in dating ancient Chinese manuscripts renders it impossible to definitively know who came first: Pythagoras or a Chinese Pythagoras. However, Greece was certainly not the only place for great mathematical discoveries. History of mathematics books would do well to write about the great Chinese and other Asian contributions so as to avoid an Indo-European ethnocentrism.

Works Consulted

Boyer, Carl B. A History of Mathematics. 2nd ed. New York: John Wiley & Sons, 1991.

Cullen, Christopher. Astronomy and Mathematics in Ancient China: the *Zhou bi suan jing*. Cambridge: Cambridge UP, 1996.

-includes a translation of *Zhou bi suan jing*

Lay-Yong, Lam, and Shen Kangsheng. "Mathematical Problems on Surveying in Ancient China." Archive for History of Exact Sciences 36.1 (1986): 1-20.

---. "Right Angled Triangles in Ancient China." Archive for History of Exact Sciences 30.2 (1984): 87-112.

Mikami, Yoshio. The Development of Mathematics in China and Japan. 2nd ed. New York: Chelsea Publishing Company, 1974.

Sweltz, Frank J. The Sea Island Mathematical Manual: Surveying and Mathematics in Ancient China. University Park: Pennsylvania State UP, 1992.

-includes a translation of *Haidao Suanjing*

Sweltz, Frank J., and T. I. Kao. Was Pythagoras Chinese? An Examination of Right Triangle Theory in Ancient China. University Park: Pennsylvania State UP, 1977.

-includes a translation of Chapter 9 of the *Chiu chang suan shu*